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ON THE TEACHING OF ANALYTIC GEOMETRY.

By A. E. YOUNG, Miama University.

The idea of the graph, the basal concept in the development of that branch of mathematics known today as analytic geometry, was a natural outgrowth, it would seem, of the line of thought undertaken by the great philosopher and mathematician Descartes; for he undertook to parallel the algebraic operations with geomatric, and was probably the first to obtain the roots of a quadratic equation in one unknown by aid of ruler and compasses. The next step, the solution of a quadratic in two unknowns, geometrically interpreted, as in the previous case, led to the coordinate idea, and hence to the graph.

Thus was born the concept which has made possible the science of mathematics as it exists today. Algebra and geometry were thus united by Descartes and others, and since their time analysis proper and geometry have continued their development hand in hand. In fact, analytic geometry may be defined as the theory of analysis geometrically interpreted. The former is therefore a dependent science, and many mathematicians thus consider it. However, for the better understanding of analysis proper, and its applications to problems of the natural sciences, it has played an important and, oftentimes, an essential role. The sciences of astronomy, physics, and mechanics are what they are today through the theory of analysis geometrically interpreted. The engineering profession is dependent upon the ideas of analytic geometry for the applications of algebra, trigonometry, calculus, differential equations, etc., to the solution of problems. The graph is as useful to the engineer as his foot-rule, and almost as constantly used in many cases. The fundamental idea of analytic geometry, that of showing graphically the relation between two (or more) variables, the change in one and the corresponding change in the other, has crept into all professions and occupations. Many lines of work are almost entirely dependent for their recent growth upon the employment of these ideas. Students return to our technical schools at the close of every summer from factories and mills with problems of loci, etc., which uneducated mechanics have given them for solution. Many men today find themselves seriously handicapped because of their lack of knowledge of this kind and the steadily increasing demand for it. This cry of the industrial world for the solution of difficult problems can not be met in most cases without the aid of analytic geometry.

In view of the foregoing conditions, therefore, we do well to consider the question of the proper method of teaching so important a subject. What should we teach and how should we teach it, are questions which may well be asked, and the answers will vary according to the experience of different teachers.

The writer has found that analytic geometry is most difficult for the beginning student. This difficulty is in part inherent in the subject, but to a great extent also it is due to the student's entire lack of previous training in this line of work and the failure of the teacher to appreciate this fact. This difficulty is being overcome somewhat, at present, by the introduction of graphical work in high school algebras, which is certainly a move in the right direction. It is to be hoped that in the near future coordinate geometry in its simpler forms may become a preparatory school subject. Of the year and a half now spent in the study of elementary algebra, one half year might well be given to the geometric interpretation of algebra, or, in other words, to the study of coordinates, graphs, simple loci problems, etc. This is desirable not only in order that the pupil may have the coordinate idea well grounded before he comes to college and goes into more advanced work, but also that the pupil who ends his course in the preparatory school, may have an opportunity to learn at least the elementary ideas of a subject which is of such general application in the everyday problems of life.

Given a class of students who have been grounded in the ideas of coordinate geometry, and granting that they have average intelligence, no other branch of mathematics is likely to be so interesting to both teacher and pupils. The work need never be dry to either. The problems are endless and can be graded so as to come within the reach of all. The student can see almost daily improvement in his own ability to grasp the ideas and solve the problems. A new field is open to him, whose vastness is revealed by a few suggestions from the teacher. No subject is so likely as this to turn the young, inexperienced student of mathematics into an enthusiast.

Before discussing the question of the subject matter, or the manner of presenting it, it is necessary to consider the class of students to be taught, why they are studying the subject, and what use they are to make of it. Let us divide them into two classes, the students of the technical school, and those of the college proper. The latter class may again be divided into those who elect the subject and those who take it as required work. In all colleges where mathematics is still a required subject, and there are a few outside of the technical schools, analytic geometry is usually required. Of course this is as it should be, though it may be doubted whether sufficient work of this nature is required in most colleges at the present time. A half year's work four times per week should be the lower limit, and even then the course might well be called "Coordinate Geometry," instead of "Analytic," leaving the latter name for a second course to be described later.

In a course of this kind, given to a class of students many of whom do no more work than they are compelled to do, the teacher meets with a hard problem, but one which is well worth solving, and probably no more difficult than that of teaching any other required subject in these days of electives. The difficulty of teaching required work at the present time is due largely to the ease with which a student can meet the requirements of the elective undergraduate work in other lines. Analytical geometry is not an easy subject compared to most of the electives. In order to make the work easier for the average student, and that the best may grasp the ideas more clearly, the teacher should spend a large part of the class-period, especially near the beginning of the course, in giving explanations. Loafing on the part of students is easily prevented by requiring them to reproduce the work done by the teacher either at the close of the recitation or at the next meeting of the class, and by frequent exercises.

In regard to the material for a class taking a short course in required work, there should be little question. Cartesian coordinates should give the first introduction to the subject, but polar coordinates should be by no means omitted. The latter are quite as much employed in some lines of practical work as the former, especially in the engineering profession.

Change of coordinates and lines of reference should be discussed till the processes become familiar to the students. Loci problems of the simpler kind should be studied carefully. These problems give great difficulty to the average student, due probably, to the awkward manner in which they are presented, in many cases. If the parametric form of the equation of a curve is considered carefully in the first place, in the case of known curves, then loci problems are easily solved by getting the equation of the curve in this form and stopping there. The elimination of the parameter (or the parameters) is an algebraic question, and one which the student should be taught to consider by itself. It is the failure to distinguish between the two steps that makes the problems difficult.

It would seem that, in general, too much attention in the past has been paid, in a course like the one under discussion, to the consideration of conics and their properties. Oftentimes the dull pupils in the class, after having finally grasped the fact that there are other loci besides straight lines, at once conclude that there must of necessity be conic sections. It would be well to introduce some of the simpler curves of the third and fourth degree, in polar coordinates, as well as cartesian, and also, some of the more common non-algebraic curves such as the catenary, spirals, and cycloids, the latter in the parametric form.

No great results can be hoped for from a course of this kind given under such conditions. If the class as a whole learn thoroughly the common facts, and are at the same time aware that they have just touched the subject itself, and the really good students have the ambition to know more about it, the teacher should be well satisfied, and doubtless would be. That this latter point may be gained it would be well for the teacher throughout the course to generalize for the benefit of the better students whenever the opportunity offers.

For example, from the polar line one may go to the polar curve, giving as a particular case the polar curves corresponding to certain cubics. The ideas of class and order may be considered just enough to arouse the curiosity of the pupils. The general method for determining asymptotes to an algebraic curve should be explained, and particular examples given. In this way we show the student some of the paths leading out, and he is naturally interested in the question as to where they lead. His ambition to know is stimulated. He begins to see how vast is the science with which he is dealing. He avoids closing the course with the idea that there were just a few points which he did not have time to learn, and he really understands that if he is to know much about the subject he must elect the course to which we now turn our attention, "elective analytic geometry," a course which should be taken up at the beginning of his Sophomore year and carried through two semesters.

We are not proposing here a course for graduate students in mathematics, but a course suited to Sophomores. In our wild attempts these days to do graduate work, we inject too much of the graduate spirit in the courses in mathematics primarily intended for the undergraduate of limited experience. The result is that the undergraduate suffers. He finds himself beyond his depth, and if he ever does touch ground before the close of the course, it is in a half drowned condition. We shall make no such mistake in outlining this course. It is taken for granted that the students who elect it have little mathematical knowledge. At most they have, or are at the same time obtaining, a little knowledge in differential calculus. To obtain the best results this should be the case.

This course should begin with a rapid review of the principal subjects considered in the previous course, namely, the straight line, the properties of conics, the transformation of coordinates, polars and tangents, etc. In connection with the study of the straight line, the idea of the point-line quality should be considered carefully and to a considerable extent. Also, line coordinates and homogeneous point coordinates should be used at the same stage of the work, and thereafter, whenever it is of advantage to do so. In connection with the idea of the tangent, the asymptote should be considered, the coordinates in use being polar as well as rectangular. The idea of the polar curve, first, second, third polar, etc., for algebraic curves should be discussed quite fully, that is, long enough to fix the ideas in the mind of the student. The cubic curves should be studied at some length. They might be divided into the five classes, cuspidal, bipartite, etc., and then the various curves in each class, characterized by the way in which they are cut by the line at infinity. No time should be spent in the careful plotting of unknown curves. Such work should be relegated to a course in calculus.

The course as outlined so far, should cover the first half of the Sophomore year and should fit the student taking it for a more advanced course in analytic geometry, which is generally called Modern Analytic Geometry. This latter course, intended for graduate students or those undergraduates who are specializing, we shall not attempt to outline here.

The great mistake, at the present time, is that of omitting a course similar to the one just outlined, and substituting in its place the one in Modern Analytic Geometry. Having taken the first, the student is infinitely better prepared for the second.

Before outlining the work of the second half year of this course, which should be in geometry of three dimensions, let us say a few words in regard to the work in analytic geometry at colleges where no mathematical work is required. Here the course should be arranged somewhat differently, we believe. The first course in analytic geometry should be made more comprehensive than when the course is required. It should include all of the work of the required course as outlined above, and some of the subjects just touched upon there should be developed more fully. A half year, four times per week, would be necessary for this work. It should be followed directly by the course in geometry of three dimensions which we are about to consider, modified somewhat to meet the new conditions.

The discussion of geometry of three dimensions should begin, of course, with a careful study of the straight line and plane, followed by work on the transformation of coordinates, change of origin and axes, etc. The discussion of the conicoids should be thorough. The general equation of the second degree should be discussed, as should also the subject of confocal conicoids. Polar and spherical coordinates should be employed in the discussion of some surfaces in which they can be used to advantage, With students who have had the required work as well as the elective course in plane analytics, the work should be done rapidly. It should be followed by an elementary discussion of space curves and algebraic surfaces. The one and two parameter idea, corresponding respectively to the curve and surface, should be touched upon sufficiently to fix the fundamental thought. The work in discussing algebraic surfaces would parallel the similar discussion of plane algebraic curves, and yield large results with little work. A class composed of students who have taken only the one course (the elective) would hardly be able to do thoroughly more than the work first outlined above.

Turning now to the teaching of analytical geometry in the Technical School, we have quite a different case for consideration. Here the classes are composed of students, the majority of whom, at least, realize that they are taking, in analytical geometry, one of the subjects which will prove most useful in their actual engineering work, and one the mastery of which is essential to success in several of the subjects which are to follow. Moreover, we can assume (though we find so many exceptions that we begin to doubt the rule) that the students have a little more mathematical knowledge when beginning the subject of analytic geometry than in the previous case.

External conditions are somewhat different from those in an academic institution. The engineering professors are demanding that the student in the technical school absorb all the mathematics that he is to have in his first year and a half, or in the first two years at most. He thus has at best little time to assimilate his mathematical food, and we do not wonder that most of it leaves only a fleeting impression. In this connection we call attention to the books which are being written purposely to meet these conditions. We refer to books on "mathematics for engineers" in which courses like calculus, analytic geometry, etc., are combined. We have no doubt but that such a condition of affairs is bad for the student with the average preparation in mathematics.

It is two much on the face of it to expect that a freshmen, prepared as ours are at present, will be able to take up a course in "engineering mathematics," in which are combined the ideas of half a dozen subjects of which he knows little or nothing, and obtain any very definite knowledge at the end of the course. "Engineering mathematics" should *follow* thorough courses in the purely mathematical subjects and should not *replace* these. Taken thus it would prove of immense benefit to the students. Taken as it is, the principles of mathematics which are so important, and a knowledge of which is so essential to the student, are in most cases not clearly understood, if grasped at all.

We do believe, however, that a quite different course in analytic geometry should be given the prospective engineer from that provided for the ordinary student. In his case it should be always kept in mind that this is a subject which he will doubtless use in a practical way ever afterward. The difficult problem which the engineer will often meet is this: Given certain conditions, what is the equation of a curve, or curves, corresponding? Or this: Given certain corresponding values of two variables represented, say by two points in a plane, to find a relation between the variables which will approximate the true relation sufficiently near for his purpose? In the writer's short experience as a teacher in an engineering school, several graduates have reported difficulty in regard to such questions. Their great mistake seemed to be their assumption that to a set of points there corresponds necessarily one curve and only one. They had not fully grasped the facts necessary to the full understanding of loci problems, nor was it to be wondered at considering the circumstances.

As of first importance, therefore, let us urge the claims of graphical work, and the great care that should be taken by the instructor to explain most carefully the theory underlying the determination of a curve which will satisfy given conditions. Many problems of the following nature should be treated: Given a certain number of points, determine whether certain algebraic curves such as the circle, the parabola, the ellipse, and certain third degree curves, pass through them. Take up cases where one arbitrary constant remains in the equation, impose another condition, etc,, etc. Consider the determination of loci whose equations hall be of some of the more common non-algebraic forms, as the catenary, spirals, or the trochoids. Consider the form of the equations of some of these if the origin and axes are changed. Be careful to see that the pupils have a great familiarity with the ideas involved in the transformation of coordinates.

The next most important part of their work should be the plotting of curves from given equations. A student who left college at the close of his junior year, reported after being out three years, engaged in construction work for one of our large railroad companies, that the study of "higher plane curves," had proved of the greatest use to him of any of his mathematical work. The ideas gained enable him to treat his problems intelligently. This sort of work, however, cannot be done without using the ideas and methods of the calculus. Partly for that reason, and partly in order that the student may begin early the study of this great subject, we would urge that no extended course be taken in analytical geometry, apart from calculus. A semester's work should enable the student to grasp the fundamental ideas of the subject of plane and solid analytic geometry. The course should then continue, but as a course in differential calculus, geometrically applied. It is a mistake to fill a book on elementary calculus with problems from all the sciences in which it may be applied. Much better would it be to confine the applications almost entirely to the science of elementary analytical geometry. In this way the student will stand a much better chance of actually grasping the principles of the calculus and will at the same time learn a great deal, and in the proper way, about a very necessary subject. It may be that when the student takes up mechanics and allied subjects, he will not start off so briskly at first but in the end he should do much better work, as he should have the principles of calculus in better shape and hence be able to apply them better. An engineer of long experience, in stating that he never made any use of calculus in his work, seemed to think that it was due to the way in which the subject had been presented to him. No doubt he was right about it, although we might disagree as to the way it should be presented. An intelligent, well-educated engineer *should* use calculus to his advantage in some of his work. In fact, the gentleman referred to above did say that engineers in the United States Government service do use it continually. A man can not use calculus understandingly unless he is familiar with the *theory* of calculus. A careful study of the theory should precede its general application, and to the fact that this is not the case in many of our technical schools, we owe the "failure of the science" as an engineering tool. It would seem that today the mathematicians are falling into this very error themselves, urged on by the professors in the applied sciences. They say to themselves, "I must not teach any pure mathematics," and they almost apologize to their classes for wasting any time on calculus itself, or analytic geometry before applying it. The result is a great deal like that of beginning the study of Greek literature before learning the Greek language. They make a mistake in this, and some of our leading educators in large engineering schools are becoming convinced of this fact. They say, "Stick to your own science. Give the students the pure mathematical ideas, and let the professors of physics, mechanics, etc., do the applying of the principles." Much better would it be to give the non-engineering student work in the broader applications of the mathematical theory, for the engineer is sure to get this work, and the other students will have no other chance.

Finally, in a course in analytic geometry for engineers, we would urge again the general use of the parametric form of the equation. It has a wide application and is of common use in their profession, as, for example, in the case of the moving point whose coordidates are functions of the time. Polar coordinates, also, are almost as familiar to the engineer in much of his work as rectangular, and should receive careful attention in the course in analytic geometry, both before and after the application of the theory of calculus.

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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

323. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that the relation $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$ can hold for real numbers only when they are in proportion.

Solution by the PROPOSER.

Expanding and rearranging, we get

$$a^2(c^2+d^2)-2abc(b+d)+b^4+b^2c^2-2bc^2d+b^2d^2+c^4=0.$$

Solving for a,

$$a=\frac{bc(b+d)\pm(c^2-bd)\sqrt{[-(b^2+c^2+d^2)]}}{c^2+d^2}$$

which can be real only when $c^2-bd=0$, *i. e.* when $c/d=b/c$. Then

$$a=\frac{bc\ (b+d)}{c^2+d^2}=\frac{bc\ (b+d)}{bd+d^2}=\frac{bc}{d},$$

since $b+d\neq0$. Therefore, $a=\frac{bc}{d}$ or $\frac{a}{b}=\frac{c}{d}=\frac{b}{c}$.

Also solved by G. B. M. Zerr.